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1969 J. Phys. A: Gen. Phys. 2 411

(http://iopscience.iop.org/0022-3689/2/3/021)

J. PHYS. A (GEN. PHYS.), 1969, SER. 2, VOL. 2. PRINTED IN GREAT BRITAIN

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Abstract. The analogue of a theorem of Van Vleck on the spectrum of clipped noise is formulated for use in optical spectroscopy.

For simple optical fields a single probability distribution of integrated-intensity fluctuations is often sufficient to determine spectral parameters (Jakeman and Pike 1968, 1969). Experimentally this method involves the determination of a photon-counting distribution $p(n, T)$, where T is the integration time (Jakeman *et al.* 1968). In the case of more complex fields, when the optical spectrum is symmetrical, many such distributions for different values of T can still be used. Equivalent information, however, is contained in the autocorrelation function of the photon-counting fluctuations which, for stationary processes, may be written $\langle n(0, T)n(\tau, T)\rangle$. This quantity is identical with the auto-correlation function of the integrated-intensity fluctuations $\langle E(0, T)E(\tau, T)\rangle$, except when $\tau = 0$, and reduces to the Fourier transform of the intensity-fluctuation spectrum when T is much smaller than any correlation time of the field. The equivalence mentioned is a consequence of the relation, which may be shown without difficulty,

$$\frac{d^2}{dT^2}\{T^2 n^{(2)}(T)\} = 2g^{(2)}(T) \tag{1}$$

where $g^{(2)}(t)$ is the normalized autocorrelation function $\langle E(0, 0)E(t, 0)\rangle/\langle E\rangle^2$ and $n^{(2)}(T)$ is the normalized second factorial moment of $p(n, T)$.

In recent years the methods of Doppler and intensity-fluctuation spectroscopy, well known in the radar field (see, for example, Atlas 1964), have, with the advent of the laser, been applied successfully in the optical region of the spectrum (Cummins *et al.* 1963, Yeh and Cummins 1964, Alpert *et al.* 1965, Ford and Benedek 1965, and many others since). Most of these measurements have been made by spectrum analysis using a scanning electrical filter, although Cummins (1968) has used an analogue correlator. In radar applications, however, more sophisticated techniques for the experimental determination of autocorrelation functions have been developed. In particular, the method of 'clipping' a fluctuating signal before correlation permits considerable simplification of the instrumentation at a small cost in experimental time. For example, Van Vleck (1943) has shown that, if a Gaussian signal is represented by unity or zero according to whether it is above or below its mean value, then the autocorrelation function of this 'hard-limited' signal is $2/\pi$ times the arc sine of the original one. We shall show that an analogous technique could prove equally useful in optical spectroscopy by deriving corresponding formulae for clipped photon-counting fluctuations. A full theoretical treatment will be presented elsewhere and we give here only a few of the more elementary results.

We shall assume throughout that T is very small and drop it from our notation, so that $\langle n(0)n(\tau)\rangle/\bar{n}^2 = g^{(2)}(\tau)$, except when $\tau = 0$. Here $\bar{n} = \langle n\rangle = \alpha\langle E\rangle$, where α is the efficiency of the detector. We also introduce the following notation for the clipped photon count:

$$\begin{aligned} n_k(t) &= 1 \text{ if } n(t) > k \\ &= 0 \text{ if } n(t) \leqslant k. \end{aligned} \tag{2}$$

Experimentally $n_k(t)$ can be measured with simpler equipment than $n(t)$. We now need to relate correlations involving $n_k(t)$ to $g^{(2)}(\tau)$. This is easily carried out for Gaussian fields, for which (Glauber 1963)

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2 \tag{3}$$

where $g^{(1)}(\tau)$ is the Fourier transform of the optical spectrum. In this case the generating function for the joint probability distribution of intensity fluctuations can be obtained by Laplace transformation of the expression given by Siegert (1943). The result is (see also Bédard 1967)

$$Q(ss') = [\bar{n}^2 ss'\{1-|g^{(1)}(\tau)|^2\}+\bar{n}(s+s')+1]^{-1} \tag{4}$$

and defines the joint photon-counting distribution as follows:

$$p\{n(0), n(\tau)\} = \frac{(-1)^{n(0)}}{n(0)!}\frac{(-1)^{n(\tau)}}{n(\tau)!}\frac{d^{n(0)}}{ds^{n(0)}}\frac{d^{n(\tau)}}{ds'^{n(\tau)}}Q(s,s')\bigg|_{s=s'=1}. \tag{5}$$

As a simple example of the use of these formulae, we consider the case of double clipping at zero photon number. The appropriate normalized correlation function is given by

$$\frac{\langle n_0(0)n_0(\tau)\rangle}{\langle n_0\rangle^2} = \frac{1}{\langle n_0\rangle^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} p(n,m) = \left(\frac{1+\bar{n}}{\bar{n}}\right)^2\{1-2p(0)+p(0,0)\}$$

$$= \left\{1+\frac{1-\bar{n}}{1+\bar{n}}|g^{(1)}(\tau)|^2\right\}\left\{1-\left(\frac{\bar{n}}{1+\bar{n}}\right)^2|g^{(1)}(\tau)|^2\right\}^{-1}. \tag{6}$$

In addition to (4) and (5), we have used the generating function for the single probability distribution of intensity fluctuations, $(1+\bar{n}s)^{-1}$. If $\bar{n} \ll 1$, (6) reduces to $g^{(2)}(\tau)$, while, if $\bar{n} \gg 1$, it approaches a constant value of unity and no spectral information is gained from this extreme form of clipping. The other example we consider in this letter is single clipping at an arbitrary photon number k. The result obtained using (4) and (5) is

$$\frac{\langle n_k(0)n(\tau)\rangle}{\langle n_k\rangle\langle n\rangle} = \frac{1}{\langle n_k\rangle\langle n\rangle}\sum_{n=1}^{\infty}\sum_{m=k+1}^{\infty} np(n,m)$$

$$= \frac{1}{\bar{n}}\left(\frac{1+\bar{n}}{\bar{n}}\right)^{k+1}\left\{\bar{n}+\sum_{m=0}^{k}\frac{(-1)^m}{m!}\frac{d^m}{ds'^m}\frac{d}{ds}Q(s-1,s')\bigg|_{s=s'=1}\right\}$$

$$= 1+\frac{1+k}{1+\bar{n}}|g^{(1)}(\tau)|^2. \tag{7}$$

In this case, for all values of $\bar{n}$, little spectral information is lost provided that clipping is carried out near the mean. If k is exactly equal to $\bar{n}$, the right-hand side again becomes $g^{(2)}(\tau)$.

The experimental use of these ideas will be discussed more fully in a future publication.

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3rd January 1969, *in revised form 27th January* 1969

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